Gradient and Laplacian in spherical coords

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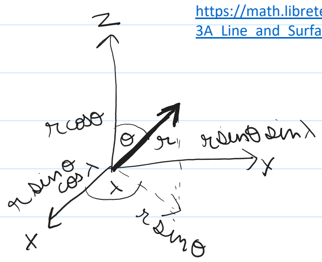
https://math.libretexts.org/Bookshelves/Calculus/Vector_Calculus_(Corral)/04%3A_Line_and_Surface_Integrals/4.06%3A_Gradient_Divergence_Curl_and_Laplacian

$f = x\hat{i} + y\hat{j} + z\hat{k}$



Spherical Coords.

$x = r\sin\theta\cos\lambda$, $y = r\sin\theta\sin\lambda$, $z = r\cos\theta$

$f = (r\sin\theta\cos\lambda)\hat{i} + (r\sin\theta\sin\lambda)\hat{j} + (r\cos\theta)\hat{k}$

$\frac{\partial f}{\partial r} = \sin\theta\cos\lambda\,\hat{i} + \sin\theta\sin\lambda\,\hat{j} + \cos\theta\,\hat{k}$

$\left|\frac{\partial f}{\partial r}\right| = \left|\sqrt{\sin^2\theta\cos^2\lambda + \sin^2\theta\sin^2\lambda + \cos^2\theta}\right| = 1$

$\frac{\partial^2 f}{\partial r^2} = 0$

$\left|\frac{\partial^2 f}{\partial \lambda^2}\right| = 0$

$\frac{\partial f}{\partial \theta} = r\cos\theta\cos\lambda\,\hat{i} + r\cos\theta\sin\lambda\,\hat{j} - r\sin\theta\,\hat{k}$

$\left|\frac{\partial f}{\partial \theta}\right| = \left|\sqrt{r^2\cos^2\theta\cos^2\lambda + r^2\cos^2\theta\sin^2\lambda + r^2\sin^2\theta}\right| = r$

$\frac{\partial^2 f}{\partial \theta^2} = -r\sin\theta\cos\lambda\,\hat{i} - r\sin\theta\sin\lambda\,\hat{j} - r\cos\theta\,\hat{k}$

$\left|\frac{\partial^2 f}{\partial \theta^2}\right| = r\,\hat{\theta}$

$\frac{\partial f}{\partial \lambda} = -r\sin\theta\sin\lambda\,\hat{i} + r\sin\theta\cos\lambda\,\hat{j}$

$\frac{\partial f}{\partial \lambda} = \left|\sqrt{r^2\sin^2\theta\sin^2\lambda + r^2\sin^2\theta\cos^2\lambda}\right| = r\sin\theta$

$\frac{\partial^2 f}{\partial \lambda^2} = -r\sin\theta\cos\lambda\,\hat{i} - r\sin\theta\sin\lambda\,\hat{j}$

$\left|\frac{\partial^2 f}{\partial \lambda^2}\right| = r\sin\theta\,\hat{\lambda}$

Gradient

f (scalar field)

Rate of max. change OR Direction of steepest ascent

$f = x\hat{i} + y\hat{j} + z\hat{k}$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$ Cartesian

$\nabla f = \frac{\frac{\partial f}{\partial r}}{\left|\frac{\partial f}{\partial r}\right|}\hat{r} + \frac{\frac{\partial f}{\partial \theta}}{\left|\frac{\partial f}{\partial \theta}\right|}\hat{\theta} + \frac{\frac{\partial f}{\partial \lambda}}{\left|\frac{\partial f}{\partial \lambda}\right|}\hat{\lambda}$

$\nabla f = \frac{\partial f}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \lambda}\hat{\lambda}$ Spherical

- **Thermodynamics (Temperature):** If the scalar field is the temperature in a room, the gradient points in the direction that gets hottest the fastest. Heat actually flows in the *opposite* direction of the gradient (from hot to cold), which is why Fourier's Law of heat conduction has a negative sign $q_v = -k\nabla T$
- **Fluid Dynamics (Pressure):** If the field represents air pressure, the gradient points toward the area of highest pressure. Wind flows from high to low pressure, so it moves in the direction of the negative gradient $f = -\nabla P$
- **Electromagnetism (Electric Potential):** If the field is voltage (electric potential), the gradient points toward higher voltage. The electric field itself is the negative gradient of the potential $E = -\nabla V$

Divergence

A (vector field)

$\nabla\cdot A = \lim_{\Delta V \to 0} \frac{1}{\Delta V}\oint_S A\cdot dS$

Net outward flux per unit Volume

Cartesian coords $A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

$\nabla\cdot A = \vec{\nabla}\cdot\vec{A} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(A_x\hat{i} + A_y\hat{j} + A_z\hat{k}\right)$

is actually

$\nabla\cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$ Cartesian

$\nabla\cdot A = \lim_{\Delta V \to 0} \frac{1}{\Delta V}\oint_S A\cdot dS$

Spherical $A = A_r\hat{r} + A_\theta\hat{\theta} + A_\lambda\hat{\lambda}$

In physics, whenever something flows or exerts a directional influence through space, divergence is the tool used to locate the "hoses" and the "drains."

1. Electromagnetism (Gauss's Law for Electricity)
The most famous example of divergence is the first of Maxwell's equations, which describes the electric field

- **What it means:** The divergence of the electric field at any point is directly proportional to the electric charge density at that point.
- **The reality:** Positive electric charges (like protons) act as "hoses" (sources) spraying out electric field lines. Negative charges (like electrons) act as "drains" (sinks) sucking in electric field lines. Empty space with no charge has zero divergence. $\nabla\cdot E = \rho/\varepsilon_0$

2. Fluid Dynamics (The Continuity Equation)
In fluid mechanics, the divergence of a fluid's velocity field multiplied by its density tells you how mass is changing over time:

$$\nabla \cdot A = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint_S A \cdot ds$$

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2. Fluid Dynamics (The Continuity Equation)
In fluid mechanics, the divergence of a fluid's velocity field multiplied by its density tells you how mass is changing over time:

- **What it means:** If the divergence of the fluid flow is positive (it's spreading out), the density of the fluid at that point must be dropping over time (hence the negative sign).
- **The reality:** If you squeeze a balloon, the air inside is compressed. If you let it expand, the air spreads out. Divergence is how aerodynamicists mathematically track whether air is compressing or expanding as it flows over an airplane wing.

$$\nabla \cdot (\rho v) = -\frac{\partial \rho}{\partial t}$$

3. Magnetism (Gauss's Law for Magnetism)
The second of Maxwell's equations is incredibly short, but it reveals a profound truth about our universe regarding the magnetic field :

- **What it means:** The divergence of a magnetic field is always, everywhere, exactly zero.
- **The reality:** There are no "magnetic hoses" or "magnetic drains." In physics terms, there are no *magnetic monopoles*. You cannot isolate a North pole or a South pole. Every magnetic field line that leaves a magnet must eventually loop back around and re-enter it. What goes out must come back in.

$$\nabla \cdot B = 0$$

Spherical coords.
$$A = A_r \hat{r} + A_\theta \hat{\theta} + A_\lambda \hat{\lambda}$$
$$dV = dx\,dy\,dz = (dr)(r\,d\theta)(r\sin\theta\, d\lambda) = r^2 \sin\theta\, d\theta\, d\lambda\, dr$$

Radial
$$\text{Area} = (r\,d\theta)(r\sin\theta\, d\lambda) = r^2 \sin\theta\, d\theta\, d\lambda$$
$$\text{Flux}_{in} = A_r \cdot r^2 \sin\theta\, d\theta\, d\lambda$$
$$\text{Net Flux}_r = \frac{\partial}{\partial r}(A_r r^2 \sin\theta)\, dr\, d\theta\, d\lambda$$

Polar
$$\text{Area} = (dr)(r\sin\theta\, d\lambda) = r\sin\theta\, dr\, d\lambda$$
$$\text{Flux}_{in} = A_\theta \cdot r\sin\theta\, dr\, d\lambda$$
$$\text{Net Flux}_\theta = \frac{\partial}{\partial \theta}(A_\theta r \sin\theta)\, d\theta\, dr\, d\lambda$$

Azimuth
$$\text{Area} = (dr)(r\,d\theta) = r\,dr\,d\theta$$
$$\text{flux}_{in} = A_\lambda \cdot r\,dr\,d\theta$$
$$\text{Net flux}_\lambda = \frac{\partial}{\partial \lambda}(A_\lambda r)\, d\lambda\, dr\, d\theta$$

$$\Rightarrow \nabla \cdot A = \frac{1}{r^2 \sin\theta\, d\theta\, d\lambda\, dr}\left(\sin\theta \frac{\partial}{\partial r}(r^2 A_r) + r\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + r\frac{\partial A_\lambda}{\partial \lambda}\right) dr\, d\theta\, d\lambda$$

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\lambda}{\partial \lambda}$$ Spherical

$\nabla^2 f < 0$ peak, $\nabla^2 f > 0$ valley, $\nabla^2 f = 0$ possibly flat or saddle

Laplacian

f (scalar field)

Divergence of gradient $\quad \nabla^2 f = \nabla \cdot (\nabla f)$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$ Cartesian

$$\nabla \cdot A = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\lambda}{\partial \lambda}$$

$$A_r = \frac{\partial f}{\partial r}, \quad A_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}, \quad A_\lambda = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \lambda}$$

$$\nabla \cdot (\nabla f) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \lambda}\left(\frac{1}{r\sin\theta}\frac{\partial f}{\partial \lambda}\right)$$

Spherical

1. The Heat Equation (Diffusion)
If you heat up one end of a metal rod, how does the temperature equalize? The heat equation dictates this process:

- **What it means:** The rate at which temperature changes over time is proportional to the Laplacian of the temperature
- **The reality:** If a specific spot is hotter than its surroundings, the equation tells us its temperature must drop over time. If a spot is colder than its surroundings (a valley), its temperature will rise. The Laplacian drives the system toward a state where there are no temperature differences

$$\frac{\partial T}{\partial t} = \alpha \nabla^2 T$$

2. The Wave Equation
Whether it's a plucked guitar string, a vibrating drumhead, or a sound wave moving through the air, waves are governed by the Laplacian:

- **What it means:** The acceleration of the wave's displacement is proportional to the Laplacian of the displacement
- **The reality:** Think of a stretched guitar string. If you pluck it, it forms a curved peak. The equation dictates that the string will experience a sharp downward acceleration to try and flatten itself out. This restoring force, driven by the curvature (Laplacian), is what causes the string to oscillate back and forth, creating a wave.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

3. Quantum Mechanics (The Schrödinger Equation)
The Laplacian we just derived in spherical coordinates is famously used to describe the hydrogen atom. In quantum mechanics, the time-independent Schrödinger equation is:

- **What it means:** The Laplacian of the electron's wavefunction represents the electron's kinetic energy.
- **The reality:** In quantum mechanics, particles behave like waves. A rapidly oscillating, highly curved wavefunction (a large Laplacian) means the particle has high momentum and high kinetic energy. A gently curving, smooth wavefunction means low kinetic energy.

$$-\frac{\hbar^2}{2m}\nabla^2 \psi + V\psi = E\psi$$

Curl

A (vector field)

$$A = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\text{curl} A = \nabla \times A \text{ or } \underset{\text{actually}}{\vec{\nabla} \times \vec{A}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\nabla \times A = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{k}$$ Cartesian

$$A = A_r \hat{r} + A_\theta \hat{\theta} + A_\lambda \hat{\lambda}$$

$$\nabla \times A = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta \, \hat{\lambda} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \lambda} \\ A_r & rA_\theta & r\sin\theta \, A_\lambda \end{vmatrix}$$

$$\nabla \times A = \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(A_\lambda \sin\theta) - \frac{\partial A_\theta}{\partial\lambda}\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\lambda} - \frac{\partial}{\partial r} r A_\lambda\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\lambda}$$

Spherical

1. **Fluid Dynamics (Vorticity)**

In fluid mechanics, the curl of the velocity field is called vorticity

- **The reality:** Vorticity describes the local spinning motion of the fluid. Think of the swirling eddies that form behind a rock in a river, the rotation of a hurricane, or the chaotic turbulence coming off an airplane wing. If a fluid has zero curl, it is called an "irrotational flow."

$$\omega = \nabla \times v$$

2. Electromagnetism (Faraday's Law of Induction)

This is one of Maxwell's equations, and it is the principle behind all electric generators.

- **What it means:** The curl of an electric field is determined by how fast the magnetic field is changing over time.
- **The reality:** If you take a magnet and plunge it through a loop of copper wire (changing the magnetic field), it induces a swirling, curling electric field inside the wire. This swirling electric field pushes the electrons in a circle, creating an electrical current.

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

3. Electromagnetism (Ampère's Law)

Another of Maxwell's equations, describing how magnetic fields are created.

- **What it means:** The curl of a magnetic field is driven by electrical current density and changing electric fields.
- **The reality:** Run a current through a straight wire, and it acts like an electromagnetic "axle." A magnetic field will instantly form, swirling in closed, curling loops around the wire.

$$\nabla \times B = \mu_0 J + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$